Crack-cluster distributions in the random fuse model

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Using large-scale numerical simulations and extensive sampling, we analyze the scaling properties of the crack-cluster distribution and the largest crack-cluster distribution at the peak load. The simulations are performed using both two-dimensional and three-dimensional random fuse models. The numerical results indicate that in contrast with the randomly diluted networks (percolation disorder), the crack-cluster distribution in the random fuse model at the peak load follows neither a power law nor an exponential distribution. The largest crack-cluster distribution at the peak load follows a lognormal distribution, and this is discussed in the context of whether there exists a relationship between the largest crack-cluster size distribution at peak load and the fracture strength distribution. Contrary to popular belief, we find that the fracture strength and the largest crack-cluster size at the peak load are uncorrelated. Indeed, quite often, the final spanning crack is formed not due to the propagation of the largest crack at the peak load, but instead due to coalescence of smaller cracks.

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I. INTRODUCTION

Fracture properties and the breakdown behavior of quasibrittle materials are very sensitive to the microstructural details of the material [1]. Indeed, in quasibrittle materials such as ceramics, the fracture strength distribution is usually dominated by the size and spatial distribution of microcracks. Traditionally, Weibull and (modified) Gumbel distributions based on the *weakest-link* approach have been widely used to describe the strength of brittle materials. These distributions naturally arise from extreme-value statistics of defect cluster distributions based on the following assumptions [2]: (1) defect clusters are independent of each other, i.e., they do not interact with one another; (2) system failure is governed by the *weakest-link* hypothesis; and (3) there exists a critical defect cluster size below which the system does not fail, and it is possible to relate the critical size of a defect cluster to the material strength. In particular, if the defect cluster size distribution is described by a power law, then the fracture strength obeys the Weibull distribution, whereas an exponential defect cluster size distribution leads to the Gumbel distribution for fracture strengths [3–6].

In this sense, crack-cluster distributions and, in particular, the largest crack-cluster distribution are of special interest due to the relation that exists between the largest crack-cluster and the fracture strength distributions [3–6]. In general, one could postulate that the size scaling and strength distribution are dictated by the statistics of the largest crack present in the system, which follow naturally from the distribution created by the damage accumulation up to the peak load.

In the case of randomly diluted disorder (percolation disorder) problems, the defect cluster size distribution is exponential far away from the percolation threshold and follows a power law close to the percolation threshold [7]. Consequently, in the randomly diluted disorder systems, in general, a Gumbel distribution better fits the fracture strengths distribution far away from the percolation threshold and a Weibull distribution provides a better fit close to the percolation threshold [3–6,8–13]. On the other hand, in broadly disordered materials, the damage accumulation is nontrivial, and is controlled by two competing aspects; namely, the disorder and the stress concentration effects around the crack-clusters. Consequently, at the peak load, the crack-cluster size distribution that evolved under the applied stress field may be quite different from the initial defect cluster size distribution of randomly diluted networks. It has been shown in Ref. [14] that neither the Weibull nor the modified Gumbel distributions may represent an adequate fit for the fracture strength distribution of broadly disordered materials.

In this regard, the relevant questions that are addressed in this paper are (i) what kind of distribution is followed by the crack-clusters in the strong disorder case just before the appearance of an unstable crack (i.e., at the peak load), and (ii) whether there exists a relationship between the distribution of the largest crack-cluster size at peak load and the fracture strength distribution?

The paper is organized as follows: in Sec. II we define the very well-studied random fuse model (RFM), which is used to study the crack-cluster distributions in both 2D triangular and 3D cubic lattice systems. In Sec. III, we report the crack-cluster distributions and the largest crack-cluster distribution for both 2D and 3D lattice systems and in Sec. IV we summarize the relation between the crack-cluster, largest crack size, and fracture strength distributions.

II. THE RANDOM FUSE MODEL

In the random thresholds fuse model [15–18], the lattice is initially fully intact with bonds having the same conduc-

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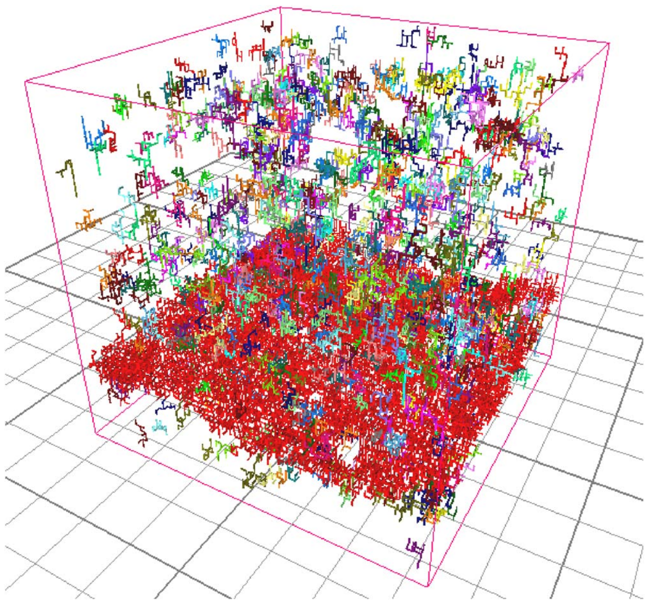

FIG. 1. (Color online) Crack-clusters in a typical simulation of a 3D cubic lattice system of size $L=64$ (close to failure). Only clusters with sizes greater than 10 are shown in the figure.

tance, but the bond breaking thresholds t are randomly distributed based on a thresholds probability distribution, $p(t)$. The burning of a fuse occurs irreversibly, whenever the electrical current in the fuse exceeds the breaking threshold current value, t, of the fuse. Periodic boundary conditions are imposed in the horizontal directions to simulate an infinite system and a constant voltage difference V is applied between the top and the bottom of the lattice system bus bars.

Numerically, a unit voltage difference $V=1$ is set between the bus bars and the Kirchhoff equations are solved to determine the current flowing in each of the fuses. Subsequently, for each fuse j, the ratio between the current i_j and the breaking threshold t_j is evaluated, and the bond j_c having the largest value, $\max_j i_j/t_j$, is irreversibly removed (burnt). The current is redistributed instantaneously after a fuse is burnt implying that the current relaxation in the lattice system is much faster than the breaking of a fuse. Each time a fuse is burnt, it is necessary to recalculate the current redistribution in the lattice to determine the subsequent breaking of a bond. The process of breaking of a bond, one at a time, is repeated until the lattice system falls apart. In this work, we assume that the bond breaking thresholds are distributed based on a uniform probability distribution, which is constant between 0 and 1.

Large-scale numerical simulations of fracture using fuse networks have been limited to smaller system sizes due to the high computational cost associated with solving a new large set of linear equations every time a new lattice bond is broken. Consequently, previous studies on scaling properties of crack-cluster distributions and their moments have been limited to numerical results obtained using smaller lattice system sizes (for example, $L=128$ in 2D) [19]. The authors have developed a rank-1 sparse Cholesky factorization downdating the algorithm for simulating fractures using discrete lattice systems [20]. In comparison with the Fourier

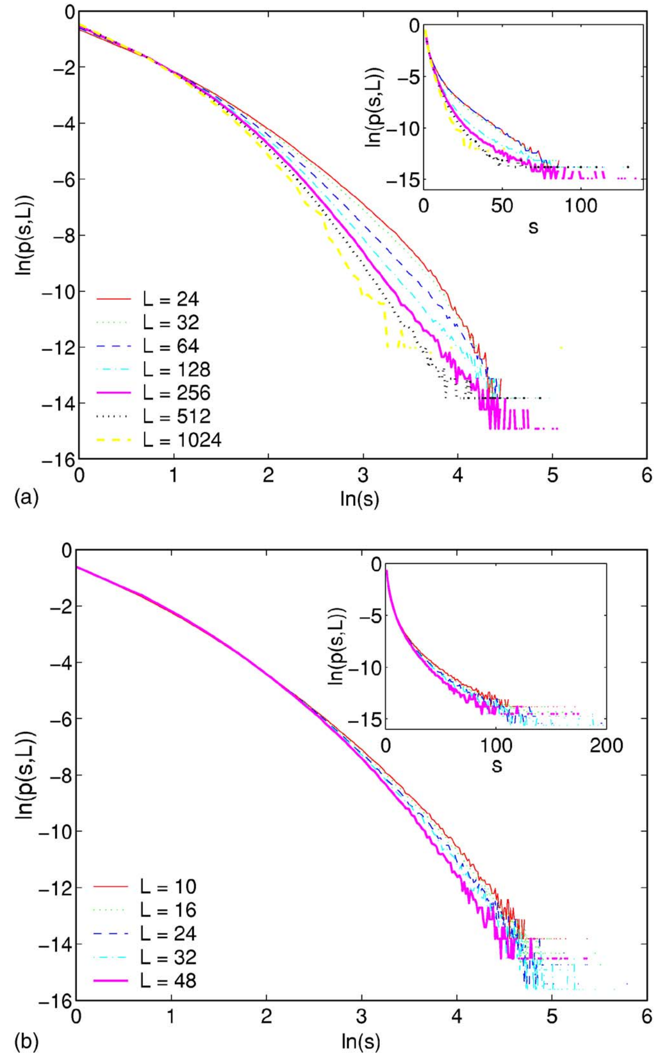

FIG. 2. (Color online) Damage cluster distribution at the peak load in 2D triangular and 3D cubic lattices. (a) 2D triangular lattice systems $L=\{24,32,64,128,256,512,1024\}$ (top). (b) 3D cubic lattice systems $L=\{10,16,24,32,48\}$ (bottom). $p(s,L)$ denotes the probability density function and s denotes the damage cluster size. Power-law fits $p(s,L)\sim\beta s^{-\beta-1}$ are shown in (a) and (b), and the corresponding exponential fits $p(s,L)\sim\beta\exp(-\beta s)$ are shown in the inset. The cluster distribution data for different lattice system sizes do not collapse on to a single straight line as it should, if the data were to follow either a power-law or an exponential distribution.

accelerated iterative schemes used in earlier studies for the modeling lattice breakdown [21], this algorithm significantly reduced the computational time required for solving large lattice systems. Using this numerical algorithm, we were able to investigate crack-clusters in larger lattice systems (e.g., $L=1024$ in 2D).

Although the sparse direct solvers presented in [20] are superior to iterative solvers for analyzing 2D lattice systems, the memory demands brought about by the amount of fill in during the sparse Cholesky factorization favor iterative solvers for analyzing 3D lattice systems. The authors have developed an algorithm based on a block-circulant preconditioned conjugate gradient (CG) iterative scheme [22] for simulating

3D random fuse networks. The block-circulant preconditioner was shown to be superior compared with the *optimal* point-circulant preconditioner for simulating 3D random fuse networks [22]. Since the block-circulant and *optimal* point-circulant preconditioners achieve favorable clustering of eigenvalues (in general, the more clustered the eigenvalues are, the faster the convergence rate is), in comparison with the Fourier accelerated iterative schemes used for the modeling lattice breakdown [23], this algorithm significantly reduced the computational time required for solving large lattice systems.

In summary, we have used the sparse Cholesky rank-1 downdating algorithm [20] for 2D lattice simulations, and the block-circulant preconditioner [22] based CG for 3D lattice simulations. For many lattice system sizes, the number of sample configurations, N_{config}, used are extremely large in order to reduce the statistical error in the numerical results. In particular, in the case of 2D RFM simulations, we have used N_{config}=50 000 for L=4,8,16,24,32,64, and N_{config}=12000,1200,200,10 for L=128,256,512,1024, respectively. Similarly, in the case of 3D RFM simulations, we used N_{config}=40000,3840,512,128,32,11 for L=10,16,24,32,48,64, respectively.

III. CRACK-CLUSTER DISTRIBUTION

In the following, the crack-cluster distribution and the largest crack distribution are examined at the peak load. Characterization of these distributions is important since these distributions determine the type of distribution followed by the fracture strength [18]. Figure 1 presents crack-clusters in a typical simulation of a 3D cubic lattice system of size L=64. As mentioned already, a power-law crack-cluster size distribution at the peak load leads to a Weibull fracture strength distribution, whereas an exponential crack-cluster size distribution at the peak load leads to a modified Gumbel-type fracture strength distribution.

Figure 2 presents the crack-cluster size distribution at the peak load in various 2D and 3D RFM lattice systems. The plots indicate that simple power-law and exponential representations are not adequate since the cluster distribution data for different lattice system sizes does not collapse on to a single straight line as it should, if the data were to follow either a power-law or an exponential distribution.

In the following, we investigate the distribution of the largest crack sizes at peak loads. Figure 3 presents the cumulative probability distribution of the largest cluster sizes at the peak load for various system sizes. The collapse of these distributions in both 2D and 3D lattices, for various system sizes, indicates that the largest crack size distribution at peak load follows a location-based probability distribution. That is, the cumulative distribution $P(s_p,L)$ of the largest cluster sizes s_p (the subscript p refers to quantities at peak load) is given by

$$P(s_p,L) = \Phi\left(\frac{s_p-\gamma}{\theta}\right) = \Phi(z), \tag{1}$$

where $z\equiv(s_p-\gamma)/\theta$, γ and θ denote the location and scale parameters of the distribution, and Φ is any cumulative probability function such that $0\leqslant\Phi\leqslant1$.

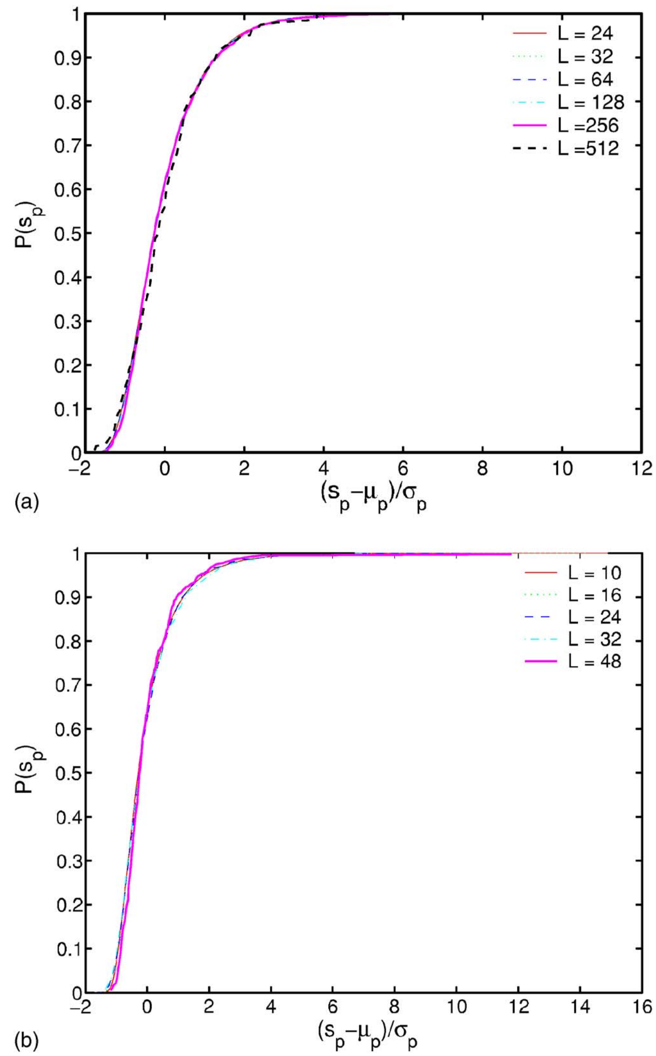

FIG. 3. (Color online) Cumulative probability distribution $P(s_p,L)$ for largest crack-cluster size s_p at peak load. (a) 2D triangular lattice systems L={24,32,64,128,256,512} (top). (b) 3D cubic lattice systems L={10,16,24,32,48} (bottom). μ_p and σ_p denote the mean and standard deviation of the largest crack size at peak load, and are related to the location and scale parameters of the probability distribution [see Eq. (1)].

The nonsymmetry of the plots in Figs. 3(a) and 3(b) clearly indicate that the largest crack-cluster distribution at the peak load is not normally distributed. On the other hand, the largest and smallest extreme value distributions (LEV and SEV, respectively) are given by

$$\Phi_{LEV}(z) = \exp[-\exp(-z)], \tag{2}$$

$$\Phi_{SEV}(z) = 1-\exp[-\exp(z)]. \tag{3}$$

The corresponding mean and standard deviations of LEV are given by $\mu_p=\gamma+\theta\nu$ and $\sigma_p^2=\theta^2\pi^2/6$, where ν=0.5772 is the Euler constant. Similarly, the mean and standard deviation of SEV are given by $\mu_p=\gamma-\theta\nu$ and $\sigma_p^2=\theta^2\pi^2/6$. It should be noted that there exists a close relationship between LEV and SEV: if a random variable $Z\sim\text{LEV}(\gamma,\theta)$, then $-Z$

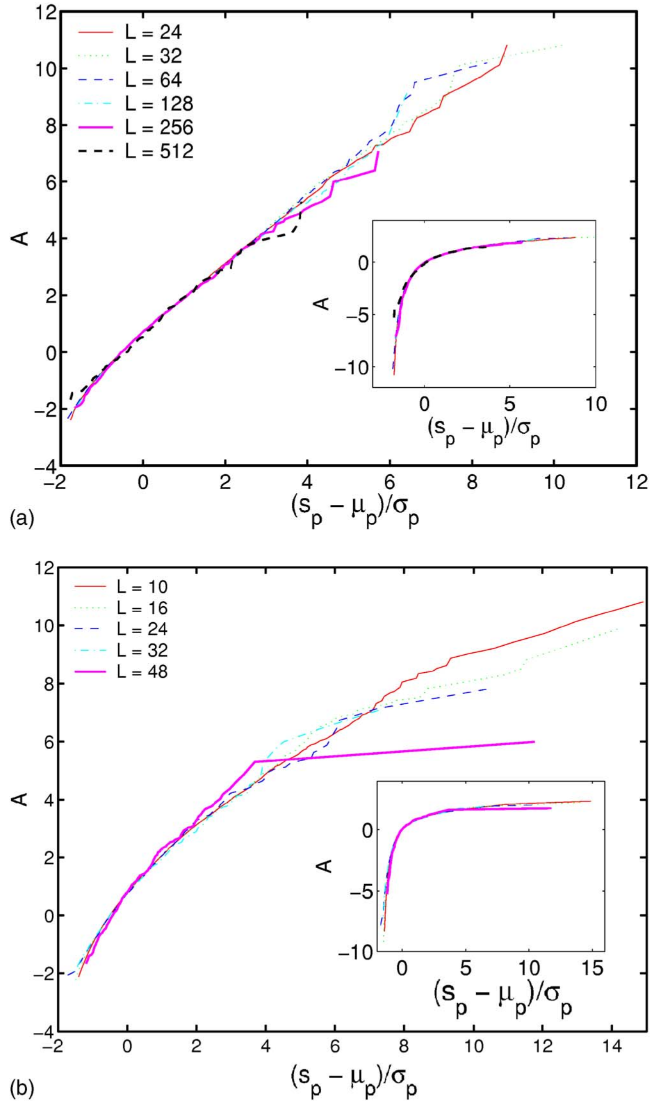

FIG. 4. (Color online) Reparametrized forms of the largest extreme value (LEV) and smallest extreme value (SEV) distibution fits for the largest crack-cluster size at the peak load. $A = -\ln[-\ln(P(s_p, L))]$ for LEV, and $A = \ln[-\ln(1-P(s_p, L))]$ for SEV distrbutions. (a) 2D triangular lattice systems $L = \{24, 32, 64, 128, 256, 512\}$ (top). (b) 3D cubic lattice systems $L = \{10, 16, 24, 32, 48\}$ (bottom). The SEV distribution fits for the data are shown in the inset.

$\sim \text{SEV}(-\gamma, \theta)$ and $\Phi^{-1}_{LEV}(p) = -\Phi^{-1}_{SEV}(1-p)$, where p refers to probability and $0 \leqslant p \leqslant 1$. Figure 4 presents the reparametrized forms LEV and SEV distribution fits for the largest crack-cluster size at peak loads. Note that in the plots of Fig. 4, we have used μ_p and σ_p instead of the location and scale parameters γ and θ. However, this only scales and shifts all of the plots in Fig. 4 equally, and hence has no effect on the collapse of the data. The collapse and linearity of the data in the reparametrized LEV representation suggest that the largest crack-cluster size distribution may adequately be described by a LEV distribution. On the other hand, significant nonlinearity can be observed in the inset of Fig. 4 indicating that the SEV distribution may not be an adequate fit for the largest crack-cluster distribution at the peak load.

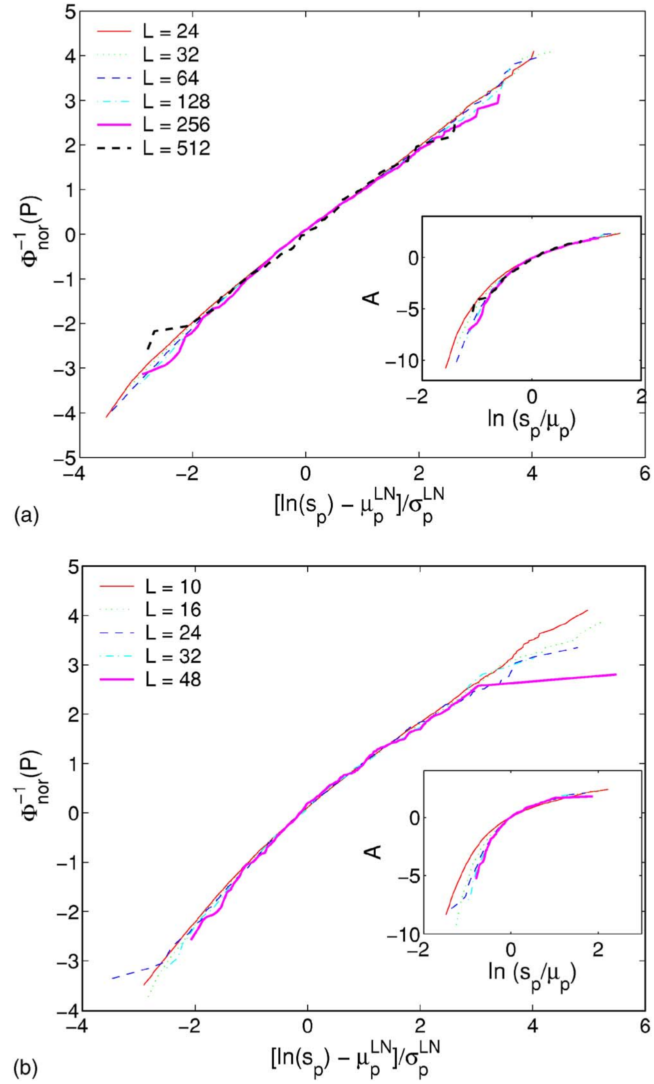

FIG. 5. (Color online) Reparametrized form of lognormal fits for the largest crack-cluster size at the peak load. (a) 2D triangular lattice systems $L = \{24, 32, 64, 128, 256, 512\}$ (top). (b) 3D cubic lattice systems $L = \{10, 16, 24, 32, 48\}$ (bottom). $\Phi_{nor}(z) = \int_{-\infty}^{z} (1/\sqrt{2\pi}) \exp(-u^2/2) du$ denotes the cumulative normal distribution function, and μ_p^{LN} and σ_p^{LN} refer to the mean and the standard deviation of the logarithm of largest cluster sizes, s_p, at peak load. The Weibull distribution fits for the data are shown in the inset $\{A = \ln[-\ln(1-P(s_p, L))]\}$. Significant nonlinearity of the plots in the inset indicates that the Weibull distribution is not an adequate fit.

On the other hand, a reparametrized lognormal distribution as shown in Fig. 5 provides an excellent fit for the largest crack-cluster distribution at the peak load. The collapse of the data as well as linearity of the plots in Fig. 5 indicate the adequacy of the lognormal distribution for representing the largest crack-cluster distribution, albeit a minute deviation from the straight line behavior can be seen in Fig. 5(b) for 3D lattice systems. In comparison, a Weibull fit as shown in the inset of Figs. 5(a) and 5(b) (a reparametrized plot) is clearly inadequate to represent the largest cluster size distribution data at the peak load. We have analyzed the adequacy of these various distributions to our data using the Akaike's information criterion [24]. These results indicate that lognor-

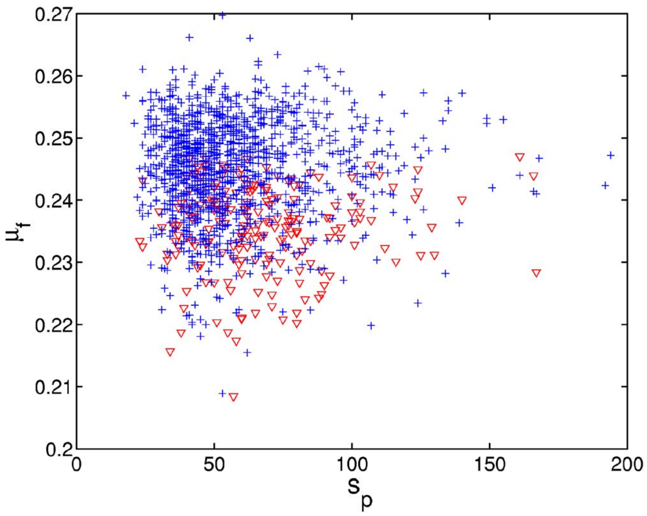

FIG. 6. (Color online) Largest crack-cluster size at the peak load versus the fracture strength. In the 2D RFM model, fracture strength is defined as the ratio of the peak current and the system size, L. The + symbols correspond to 2D RFM data with $L=256$ and the triangles correspond to 2D RFM data with $L=512$. Clearly, no or very small correlation exists between the largest cluster size at the peak load and the peak load (or fracture strength).

mal distribution (shown in Fig. 5) is a better fit compared with the LEV distribution fit shown in Fig. 4. We have also analyzed logistic and log-logistic distributions to fit the largest crack-cluster distribution; however, these distributions do not fit the data as well as the lognormal distribution does.

It is interesting to note that in the case of broadly disordered materials, the fracture strength distribution also follows a lognormal distribution [14]. Assuming that the stress concentration around the largest crack-cluster is given by a power law (with any power exponent as it is irrelevant what the value of the exponent is), it could be argued that the the largest crack-cluster size distribution and the fracture strength distribution are consistent with one another since the power of a lognormal distribution is also a lognormal distribution. However, the data in Fig. 6, wherein the largest crack-cluster size at the peak load is plotted against the corresponding fracture strength, clearly show that the largest crack-cluster size at the peak load and the fracture strength are uncorrelated. In many of the simulations, the final spanning crack is formed due to coalescence of smaller cracks and not due to the propagation of the largest crack at peak loads (see Fig. 7). Indeed, out of 200 samples of 2D RFM simulations of system size $L=512$, only 89 times the largest crack at peak loads propagated into a spanning crack at failure, whereas in all other cases, the final spanning crack is formed due to the coalescence of smaller cracks. This probably explains why the largest crack-cluster size at the peak load is not correlated with the fracture strength.

IV. CONCLUSIONS

In this paper, we have tried to address two relevant questions pertaining to crack-cluster distributions in broadly disordered materials. First, what kind of distribution is followed

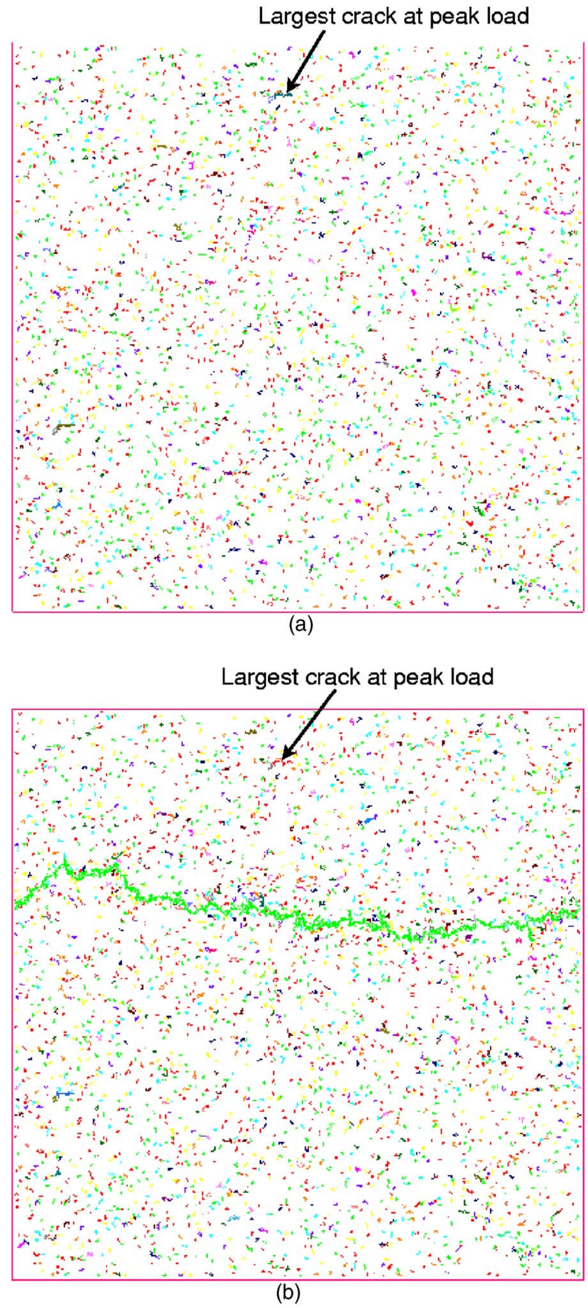

FIG. 7. (Color online) Crack-clusters at peak loads (top) and at failures (bottom) in a 2D RFM simulation of system size $L=512$. The figures show all crack-clusters whose size is greater than or equal to 5. The largest crack at the peak load is shown by the arrow. The final crack at failure is formed due to coalescence of smaller cracks, and not due to the propagation of the largest crack at the peak load.

by crack-clusters just before the appearance of an unstable crack (i.e., at the peak load), and second, whether there exists a relationship between the distribution of the largest crack-cluster size at peak loads and the fracture strength distribution?

To this end, we have analyzed the crack-cluster distribution and the largest crack-cluster size distribution at the peak load using both 2D and 3D random fuse models. The numerical results suggest that the crack-cluster distribution obtained at the peak load in the random fuse models is different from that obtained in randomly diluted networks (percolation disorder) in the sense that neither a power law nor an exponential distribution represents an adequate fit. The largest crack-cluster distribution follows better with a lognormal distribution. Although at first thought, this lognormal distribution of largest crack-cluster size at the peak load appears to be consistent with the lognormal distribution of fracture strengths as proposed in Ref. [14] (if one assumes that the stress concentration around the largest crack-cluster is given by a power law), a close examination reveals that the largest crack-cluster size and the fracture strength data are uncorrelated.

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